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Spherical harmonic representation of the electromagnetic field produced by a moving pulse of current density

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Abstract. Representation of a transient electromagnetic field generated by a pulsed current moving with a uniform velocity in terms of modes in the spherical coordinates is considered. Peculiarities of the spacetime structure of these modes in relation to the observation location and time as well as to the source current pulse duration, velocity of the pulse front, and the radiator's length are investigated. Possibilities of an adequate description of the fields due to the above source are discussed.

1. Introduction

The goal of the present paper is to construct the axisymmetric transient solution in terms of the spherical harmonics to the inhomogeneous Maxwell equations. The source is a moving pulsed radial current starting at a fixed time and moving with a constant velocity. We discuss the peculiarities of application of this solution to the description of the electromagnetic wave produced by the above source. The solution of the electrodynamic problem is derived in the spacetime domain using the method described in [1]:

(1) The electric and magnetic field vectors are expressed in terms of one scalar function that reduces the vector problem to the scalar one.

(2) The solution of the scalar problem is constructed by means of the Smirnov method of incomplete separation of variables [2]. Separating the polar-angle variable we get the solution as the Legendre polynomial series whose coefficients, being functions of the radial and time variables, satisfy the Euler–Poisson–Darbou equation.

(3) The analytical expressions for the above coefficients are obtained with the help of the Riemann formula.

Having obtained the solution of the scalar problem, one can find the non-zero component of the magnetic field by differentiation with respect to the polar angle. To obtain the components of the electric field, we have to integrate the scalar solution over the time variable. This representation of the electromagnetic field is, in fact, its expansion in terms of the spherical harmonics.

Expansions of the transient electromagnetic field in terms of the spherical harmonics were first constructed in [3–5] from the retarded Hertz vector [6–8] with the addition theorem

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for the spherical harmonics [9] and integral theorems [10]. Then the components of the electromagnetic field are given by the Hertz vectors or by the vector and scalar potentials. However, for the above special case of a moving pulsed current, it is convenient to construct the solution of Maxwell's equations by the general method [1]. The reasons to write this article are the following:

(i) The well known expansions in terms of the spherical harmonics are obtained for various steady-state (time independent or sinusoidal) fields while the explicit relations for the transient fields are reported just for some individual cases [1, 11–13].

(ii) The moving pulse is a special case of the spacetime distribution of the source current—here the spacetime structure of the emitted wave is particularly complicated and the electromagnetic field produced by the above source pulse has specific properties (especially for the case of pulse velocity equal to the velocity of light).

(iii) Application of the general expressions derived in [1, 3–5] to the description of the above transient fields requires careful preliminary consideration of its feasibility (see section 6).

2. Basic relations

In the spherical coordinates r, ϑ, φ whose origin coincides with the starting point of the axisymmetric radial current pulse moving with a constant velocity V, the current density vector has only one non-zero component:

$$\begin{aligned} &\boldsymbol{j} = j_r \boldsymbol{e}_r \\ &j_r = \frac{1}{2\pi r^2} h(\beta\tau - r) h(r - \beta(\tau - T)) h(l - r) F(\tau, r, \vartheta) \qquad \tau > 0 \\ &j_r \equiv 0 \qquad \tau < 0. \end{aligned} \tag{1}$$

Here $\boldsymbol{e}_r$ is the unit radial vector,

$$h(s) = \begin{cases} 1 & \text{for } s > 0 \\ 0 & \text{for } s < 0 \end{cases}$$

is the Heaviside step function, $\tau = ct$ is the time variable (t is time and c is the velocity of light), $\beta = V/c$ is dimensionless velocity ($0 < \beta \leqslant 1$), T is the pulse duration. We choose the constant parameter l as the minimum value for which the area of current distribution is confined by a spherical domain of radius l for all moments of time (finite radiator). For various problems the residual part of the current term $F(\tau, r, \vartheta)$ can be represented in the form $F(\tau, r, \vartheta) = F(\vartheta) f(\tau, r)$ where $F(\vartheta)$ is the angular distribution. A simple example of this distribution is $F(\vartheta) = \cos\vartheta$. For the source pulse moving along a straight line $F(\vartheta) = \delta(\cos\vartheta - 1)$ where δ is the Dirac delta-function.

Owing to the choice of the coordinate system and the axial symmetry one can obtain from Maxwell's equations for free space

$$\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}(r D_\vartheta) - \frac{1}{r}\frac{\partial D_r}{\partial \vartheta} = -\frac{1}{c}\frac{\partial H_\varphi}{\partial \tau} \\ &-\frac{1}{r}\frac{\partial}{\partial r}(r H_\varphi) = c\frac{\partial D_\vartheta}{\partial \tau} \\ &\frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(H_\varphi \sin\vartheta) = c\frac{\partial D_r}{\partial \tau} + j_r. \end{aligned} \tag{2}$$

Here we use SI units; the components of the electric induction and the magnetic field strength vectors are D_r, D_ϑ, and H_φ.

The initial conditions are

$$D_r = D_\vartheta \equiv 0 \qquad H_\varphi \equiv 0 \qquad \tau < 0. \tag{3}$$

There are several possibilities to describe the electromagnetic field with the help of two scalar functions (potentials), see [14] and also [8] for details. For this case the electromagnetic field components can be expressed via one-component radial Hertz vector $\mathbf{\Pi} = \Pi(\tau, r, \vartheta)\boldsymbol{e}_r$ introduced by Debye [15] and Bromwich [16] (see [14] for an extended consideration)

$$D_r = -\frac{\partial^2\Pi}{\partial\tau^2} + \frac{\partial^2\Pi}{\partial r^2} \qquad D_\vartheta = \frac{1}{r}\frac{\partial^2\Pi}{\partial r\partial\vartheta} \qquad H_\varphi = -\frac{c}{r}\frac{\partial^2\Pi}{\partial\tau\partial\vartheta}. \tag{4}$$

Equation (2) together with initial conditions (3) yield the scalar problem

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)\right)\frac{\partial\Pi}{\partial\tau} = \frac{1}{c}j_r \qquad \frac{\partial\Pi}{\partial\tau} \equiv 0 \qquad \tau < 0. \tag{5}$$

As far as $r \in (0, \infty)$, one needs a boundary condition at the limiting point $r = 0+$. Let us suppose that

$$\left.\frac{\partial\Pi}{\partial\tau}\right|_{r=0+} = 0. \tag{6}$$

Here we require, in fact, that the magnetic field strength obeys the condition

$$\left.rH_\varphi\right|_{r=0+} = 0. \tag{7}$$

Representing the scalar functions $\partial\Pi/\partial\tau$ and j_r in terms of the Legendre polynomials $P_n(\cos\vartheta)$,

$$\frac{\partial\Pi}{\partial\tau} = \sum_{n=0}^{\infty}\frac{\partial\Pi_n}{\partial\tau}P_n(\cos\vartheta) \qquad j_r = \sum_{n=0}^{\infty} j_n P_n(\cos\vartheta) \tag{8}$$

we separate the polar-angle variable in (5) and obtain the problem for the expansion coefficients $\partial\Pi_n/\partial\tau$

$$\begin{gathered}\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial r^2} + \frac{n(n+1)}{r^2}\right)\frac{\partial\Pi_n(\tau, r)}{\partial\tau} = \frac{1}{c}j_n(\tau, r) \\ \frac{\partial\Pi_n}{\partial\tau} \equiv 0 \qquad \tau < 0 \qquad \left.\frac{\partial\Pi_n}{\partial\tau}\right|_{r=0+} = 0\end{gathered} \tag{9}$$

with the expansion coefficients for the current density

$$j_n(\tau, r) = \frac{1}{2\pi r^2}h(\beta\tau - r)h(r - \beta(\tau - T))h(l - r)F_n(\tau, r) \tag{10}$$

where F_n are coefficients of the representation

$$F(\tau, r, \vartheta) = \sum_{n=0}^{\infty} F_n(\tau, r)P_n(\cos\vartheta). \tag{11}$$

One can find the coefficients $\partial\Pi_n/\partial\tau$ from the problem (9) with the help of the Riemann formula

$$\frac{\partial\Pi_n}{\partial\tau} = \frac{1}{2c}\iint_{\mathcal{D}} \mathrm{d}\tau'\,\mathrm{d}r'\,j_n(\tau', r')P_n\left(\frac{r^2 + r'^2 - (\tau - \tau')^2}{2rr'}\right) = \frac{1}{4\pi c}I_n(\tau, r) \tag{12}$$

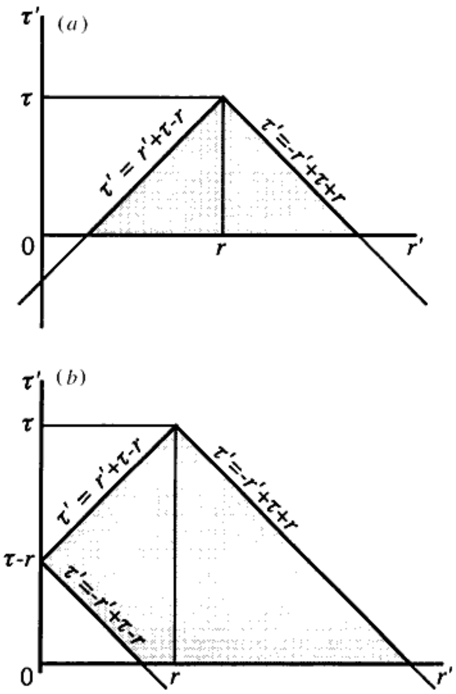

Figure 1. Integration domain to the solution of problem (9) on the r', τ' plane.

where

$$
I_n(\tau, r) = \iint_{\mathcal{D}} \mathrm{d}\tau' \,\mathrm{d}r' h(\beta\tau' - r') h(r' - \beta(\tau' - T)) h(l - r') \\
\times \frac{F_n(\tau', r')}{r'^2} P_n\left(\frac{r^2 + r'^2 - (\tau - \tau')^2}{2rr'}\right). \tag{13}
$$

The integration domain $\mathcal{D}$ is shown in figure 1(a) for $\tau < r$ and in figure 1(b) for $\tau > r$. The solution to the scalar problem (5), (6) is constructed from (8), (11), and (12):

$$
\frac{\partial \Pi}{\partial \tau}(\tau, r, \vartheta) = \frac{1}{4\pi c} \sum_{n=0}^{\infty} I_n(\tau, r) P_n(\cos\vartheta). \tag{14}
$$

Having obtained the solution to the scalar problem, one can readily obtain, with the help of (4), the expansion of the magnetic field strength in terms of the non-steady-state multipole fields

$$
H_\varphi = \sum_{n=0}^{\infty} H_{\varphi n} = -\frac{c}{r} \sum_{n=0}^{\infty} \frac{\partial \Pi_n}{\partial \tau} \frac{\partial}{\partial \vartheta} P_n(\cos\vartheta) = -\frac{1}{4\pi r} \sum_{n=1}^{\infty} I_n(\tau, r) P_n^1(\cos\vartheta) \tag{15}
$$

where $P_n^1(\cos\vartheta) \stackrel{\mathrm{def}}{=} (\partial/\partial\vartheta) P_n(\cos\vartheta)$. To obtain the components of the electric induction, one has to integrate (14) with respect to the time variable, which is not, in general, a trivial task.

3. Current pulse of infinite duration

In this section we introduce the general expression for description of the magnetic field produced by the current pulse of infinite duration ($T \to \infty$) moving within a bounded region of space of the radius l

$$j_r = \frac{1}{2\pi r^2} h(\beta\tau - r) h(l - r) F(\tau, r, \vartheta).$$

Here each member of expansion (15) has the form

$$H_{\varphi n} = -\frac{1}{4\pi r} I_n P_n^1(\cos\vartheta) = -\frac{1}{4\pi r} P_n^1(\cos\vartheta) \iint_{\mathcal{D}} \mathrm{d}\tau'\, \mathrm{d}r'\, h(\beta\tau' - r') h(l - r') \times \frac{F_n(\tau', r')}{r'^2} P_n\left(\frac{r^2 + r'^2 - (\tau - \tau')^2}{2rr'}\right). \tag{16}$$

Due to the step functions in the integrand, in most cases the actual integration domain differs from the domain $\mathcal{D}$ shown in figures 1(a) and (b). Thus, different limits of integration should be used for I_n in (16) depending on the interrelations between τ, r, β, and l. For the case $r > l$, when the observation point lies outside source's region, all possible domains of integration are shown in figures 2(a)–(c):

(a.i) If $0 < \tau - r < \frac{1-\beta}{\beta} l$ (see figure 2(a)), one has

$$I_n = \int_0^{\frac{\beta}{1+\beta}(\tau - r)} \mathrm{d}r' \int_{-r'+\tau-r}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r') + \int_{\frac{\beta}{1+\beta}(\tau-r)}^{\frac{\beta}{1-\beta}(\tau-r)} \mathrm{d}r' \int_{\frac{r'}{\beta}}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r') \tag{17}$$

where

$$\Phi_n(\tau', r') = \frac{F_n(\tau', r')}{r'^2} P_n\left(\frac{r^2 + r'^2 - (\tau - \tau')^2}{2rr'}\right) \tag{18}$$

while $\tau - r$ is the observation time reckoned from the moment of the wavefront arrival at the observation point (r, θ, φ). Evidently, for all previous moments of time, $\tau - r < 0$, one has $I_n(\tau, r) \equiv 0$. From here on we will consider positive values of $\tau - r$ only.

(a.ii) For the case $\frac{1-\beta}{\beta} l < \tau - r < \frac{1+\beta}{\beta} l$, which is illustrated in figure 2(b)

$$I_n = \int_0^{\frac{\beta}{1+\beta}(\tau - r)} \mathrm{d}r' \int_{-r'+\tau-r}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r') + \int_{\frac{\beta}{1+\beta}(\tau-r)}^{l} \mathrm{d}r' \int_{\frac{r'}{\beta}}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r'). \tag{19}$$

(a.iii) If $\tau - r > \frac{1+\beta}{\beta} l$, see figure 2($c$),

$$I_n = \int_0^{l} \mathrm{d}r' \int_{-r'+\tau-r}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r'). \tag{20}$$

To obtain the total solution for given location, one should successively use formulae (17)–(20).

For the case $r < l$ we have another set of parameter interrelations governing the type of the expression for I_n (and, consequently, for H_φ, D_r, and D_ϑ) and another set of integration domains (see figure 3):

(b.i) $r < \tau < r/\beta$, which is equivalent to $\tau + r < \frac{1+\beta}{\beta} r$ together with $\tau + r < \frac{1+\beta}{\beta} l$ (note that $\tau + r$ corresponds to the argument of the wave propagating towards the origin of coordinates), in this case one has the same triangle integration domain as in case (a.i));

(b.ii) $r/\beta < \tau < l/\beta + l - r$, which can be reduced to $\frac{1+\beta}{\beta} r < \tau + r < \frac{1+\beta}{\beta} l$;

(b.iii) $l/\beta + l - r < \tau < l/\beta + l + r$, which can be rewritten as $\tau - r < \frac{1+\beta}{\beta} l < \tau + r$; and

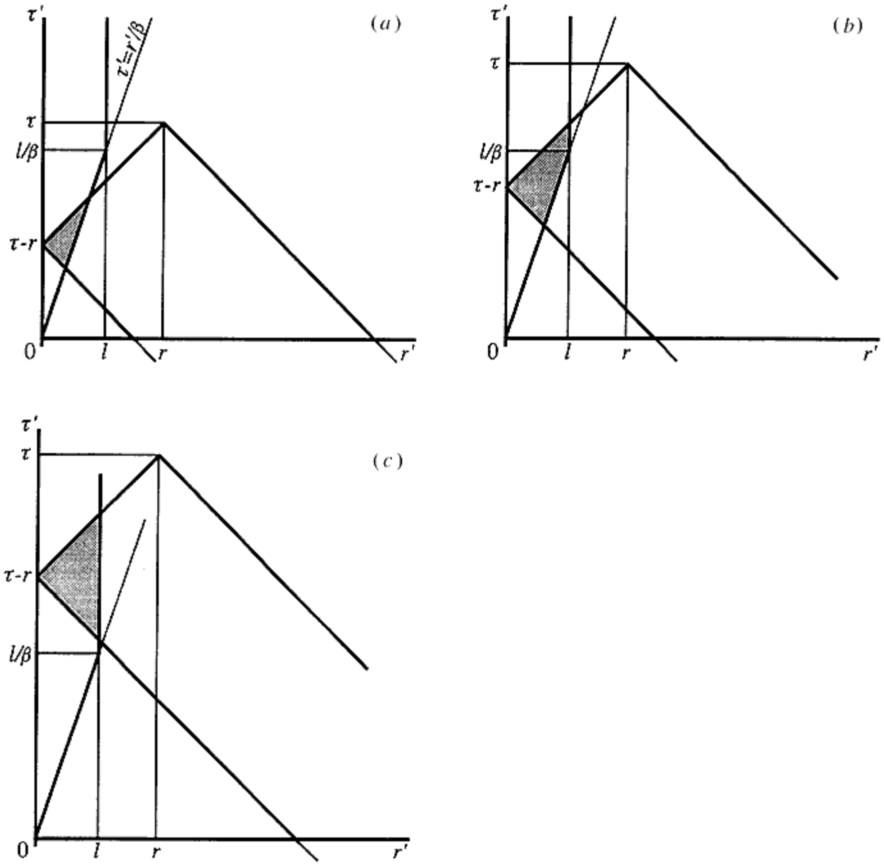

Figure 2. The r', τ'-plane diagrams for the current pulse of infinite duration ($r > l$): (a) case (a.i), (b) case (a.ii), and (c) case (a.iii).

(b.iv) $\tau > l/\beta + l + r$, which is equivalent to $\tau - r > \frac{1+\beta}{\beta} l$.

Using diagrams on the r', τ'-plane, one can easily obtain the limits of integration to each of the above instances. For example, in case (b.iv)

$$I_n = \int_0^r \mathrm{d}r' \int_{-r'+\tau-r}^{r'+\tau-r} \mathrm{d}\tau' \, \Phi_n(\tau', r') + \int_r^l \mathrm{d}r' \int_{-r'+\tau-r}^{-r'+\tau+r} \mathrm{d}\tau' \, \Phi_n(\tau', r'). \qquad (21)$$

Note that cases (b.i), (b.ii), and (b.iv) lead to two-term expressions while the most complicated case (b.iii) results in three terms. Simpler formulae can be obtained if we turn to the variables $\xi'_{1,2} = \tau' \mp r'$ and $\xi_{1,2} = \tau \mp r$. For this representation the initial expression for I_n (13) becomes

$$I_n(\xi_1, \xi_2) = 2 \iint_{\mathcal{D}} \mathrm{d}\xi'_1 \, \mathrm{d}\xi'_2 h\left(\frac{1+\beta}{1-\beta}\xi'_1 - \xi'_2\right) h\left(\xi'_2 - \frac{1+\beta}{1-\beta}\xi'_1 + \frac{2\beta}{1-\beta}T\right)$$
$$\times h(\xi'_1 + 2l - \xi'_2) \frac{F_n(\xi'_1, \xi'_2)}{(\xi'_2 - \xi'_1)^2} P_n\left(1 - 2\frac{(\xi_2 - \xi'_2)(\xi_1 - \xi'_1)}{(\xi'_2 - \xi'_1)(\xi_2 - \xi_1)}\right). \qquad (22)$$

Integration domains to cases (b.i)–(b.iv) on the ξ'_1, ξ'_2-plane are shown in figure 4.

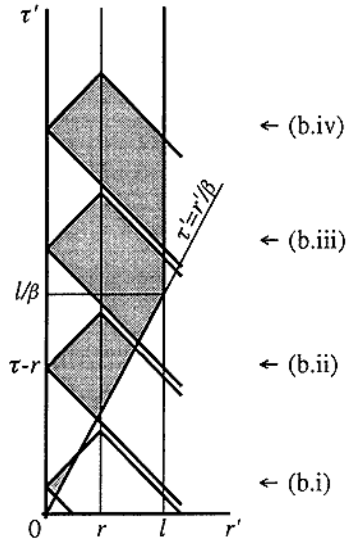

Figure 3. The r', τ'-plane diagrams for a current pulse of infinite duration, instance $r < l$.

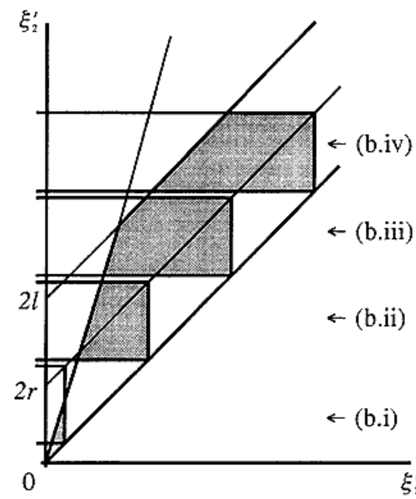

Figure 4. The ξ_1', ξ_2'-plane diagrams for a current pulse of infinite duration, instance $r < l$.

4. Current pulse of finite duration

For the case of a finite current-pulse duration the expressions for the field components can be obtained (1) using the results of the above section and the principle of field superposition or (2) constructing directly the integration domains for the current density

$$j_r = \frac{1}{2\pi r^2} h(\beta\tau - r) h(r - \beta(\tau - T)) h(l - r) F(\tau, r, \vartheta).$$

To use the first method one should rewrite the above formula in the form

$$j_r = \frac{1}{2\pi r^2} h(\beta\tau - r) h(l - r) F(\tau, r, \vartheta)$$
$$-\frac{1}{2\pi r^2} h(\beta(\tau - T) - r) h(l - r) F(\tau, r, \vartheta) \stackrel{\text{def}}{=} j_r^{(1)} + j_r^{(2)}$$

which indicate explicitly that the electromagnetic field produced by a source pulse of finite duration may be described as a sum of the fields produced by two pulses of infinite duration. The analytical expressions for the nth term of the field series determined by the second source can be obtained from the results of the previous section by using the variable $\tilde{\tau} = \tau - T$ instead of τ. Here

$$\begin{aligned} j_r^{(2)} &= -\frac{1}{2\pi r^2} h(\beta\tilde{\tau} - r) h(l - r) F(\tilde{\tau} + T, r, \vartheta) = \sum_{n=0}^{\infty} j_n^{(2)} P_n(\cos\vartheta) \\ j_n^{(2)}(\tilde{\tau}, r) &= -\frac{1}{2\pi r^2} h(\beta\tilde{\tau} - r) h(l - r) F_n(\tilde{\tau} + T, r) \end{aligned} \tag{23}$$

and for $\partial\Pi_n/\partial\tau$ one has the problem which is analogous to (9)

$$\begin{aligned} &\left(\frac{\partial^2}{\partial\tilde{\tau}^2} - \frac{\partial^2}{\partial r^2} - \frac{n(n+1)}{r^2}\right) \frac{\partial\Pi_n(\tilde{\tau}, r)}{\partial\tilde{\tau}} = \frac{1}{c} j_n^{(2)}(\tilde{\tau}, r) \\ &\frac{\partial\Pi_n}{\partial\tilde{\tau}} \equiv 0 \qquad \tilde{\tau} < 0 \qquad \left.\frac{\partial\Pi_n}{\partial\tilde{\tau}}\right|_{r=0+} = 0. \end{aligned} \tag{24}$$

The solution to this problem is given by (12) where the time variable τ is replaced by $\tilde{\tau}$ and can be easily obtained in the explicit form with the use of the r', τ'-plane diagrams of figure 3 or ξ_1', ξ_2'-plane diagrams of figure 4. Then one can construct the nth term

$$H_{\varphi n}^{(2)}(\tau, r) = -\frac{1}{4\pi r} I_n^{(2)}(\tau, r) P_n^1(\cos\vartheta)$$

and obtain the total magnetic field of the second source with the help of (15).

For $r > l$ one can use the results (17)–(20) for appropriate parameter interrelations. For example, case (a.i) corresponds to the inequality

$$0 < \tilde{\tau} - r < \frac{1-\beta}{\beta} l \tag{25}$$

(that is $T < \tau - r < T + \frac{1-\beta}{\beta} l$) and to the explicit solution

$$I_n^{(2)} = -\int_0^{\frac{\beta}{1+\beta}(\tau-T-r)} \mathrm{d}r' \int_{-r'+\tau-T-r}^{r'+\tau-T-r} \mathrm{d}\tau'\, \Phi_n^{(2)}(\tau', r') \\ - \int_{\frac{\beta}{1+\beta}(\tau-T-r)}^{\frac{\beta}{1-\beta}(\tau-T-r)} \mathrm{d}r' \int_{\frac{r'}{\beta}}^{r'+\tau-T-r} \mathrm{d}\tau'\, \Phi_n^{(2)}(\tau', r').$$

Due to the form of F_n in the inhomogeneous term (23) and the replacement $\tau \to \tilde{\tau}$ in the Riemann function, here $\Phi_n^{(2)}$ should be written as

$$\Phi_n^{(2)}(\tau', r') = \frac{F_n(\tau' + T, r')}{r'^2} P_n\left(\frac{r^2 + r'^2 - (\tau - T - \tau')^2}{2rr'}\right)$$

which is equal to $\Phi_n(\tau' + T, r')$ from the previous section. So, the expression for $I_n^{(2)}$ can be reduced to

$$\begin{aligned} I_n^{(2)} = &-\int_0^{\frac{\beta}{1+\beta}(\tau-T-r)} \mathrm{d}r' \int_{-r'+\tau-T-r}^{r'+\tau-T-r} \mathrm{d}\tau'\, \Phi_n(\tau' + T, r') \\ &- \int_{\frac{\beta}{1+\beta}(\tau-T-r)}^{\frac{\beta}{1-\beta}(\tau-T-r)} \mathrm{d}r' \int_{\frac{r'}{\beta}}^{r'+\tau-T-r} \mathrm{d}\tau'\, \Phi_n(\tau' + T, r') \\ = &-\int_0^{\frac{\beta}{1+\beta}(\tau-T-r)} \mathrm{d}r' \int_{-r'+\tau-r}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r') \\ &- \int_{\frac{\beta}{1+\beta}(\tau-T-r)}^{\frac{\beta}{1-\beta}(\tau-T-r)} \mathrm{d}r' \int_{\frac{r'}{\beta}+T}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r'). \end{aligned} \tag{26}$$

If, in addition, $0 < \tau - r < \frac{1-\beta}{\beta} l$, that together with inequality (25) yields $T < \tau - r < \frac{1-\beta}{\beta} l$, one has case (a.i) for the first term $I_n^{(1)}$ and modified case (a.i) for the second term $I_n^{(2)}$. Thus, here the total field is defined by the integral

$$\begin{aligned} I_n = I_n^{(1)} + I_n^{(2)} = &\int_0^{\frac{\beta}{1+\beta}(\tau-r)} \mathrm{d}r' \int_{-r'+\tau-r}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r') \\ &+ \int_{\frac{\beta}{1+\beta}(\tau-r)}^{\frac{\beta}{1-\beta}(\tau-r)} \mathrm{d}r' \int_{\frac{r'}{\beta}}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r') \\ &- \int_0^{\frac{\beta}{1+\beta}(\tau-T-r)} \mathrm{d}r' \int_{-r'+\tau-r}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r') \\ &- \int_{\frac{\beta}{1+\beta}(\tau-T-r)}^{\frac{\beta}{1-\beta}(\tau-T-r)} \mathrm{d}r' \int_{\frac{r'}{\beta}+T}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r') \end{aligned}$$

$$= \int_{\frac{\beta}{1+\beta}(\tau-T-r)}^{\frac{\beta}{1+\beta}(\tau-r)} \mathrm{d}r' \int_{-r'+\tau-r}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r')$$
$$+ \int_{\frac{\beta}{1+\beta}(\tau-r)}^{\frac{\beta}{1-\beta}(\tau-r)} \mathrm{d}r' \int_{\frac{r'}{\beta}}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r')$$
$$- \int_{\frac{\beta}{1+\beta}(\tau-T-r)}^{\frac{\beta}{1-\beta}(\tau-T-r)} \mathrm{d}r' \int_{\frac{r'}{\beta}+T}^{r'+\tau-r} \mathrm{d}\tau'\, \Phi_n(\tau', r') \tag{27}$$

where the function $\Phi_n(\tau', r')$ is defined by (18).

Evidently, the structure of the nth term of the electromagnetic field expansion cannot be described by a unified formula. As in the case of the source-current pulse of infinite duration, here the limits of integration depend on the interrelations between the characteristics of the pulse l, β, and T and the observation conditions τ and r. These interrelations are more complicated than those of section 3 due to the additional parameter T.

We can use another approach to the problem and construct the integration domains directly to solution (12) for the current density in its initial form (1). Let us again begin with the case $r > l$. In addition, we consider the instance $T > \frac{1+\beta}{\beta} l$ that corresponds to the so-called 'long' pulse. Corresponding domains of integration for different values of the observation time reckoned from the moment of the wavefront arrival at the observation point, $\tau - r$, are shown in figure 5. Inter-relations between parameters can be subdivided into the following cases:

(c.i) For $0 < \tau - r < \frac{1-\beta}{\beta} l$ the value of I_n is defined by relation (17) to the case (a.i) of section 3.

(c.ii) If $\frac{1-\beta}{\beta} l < \tau - r < \frac{1+\beta}{\beta} l$, the integral I_n is described by formula (19).

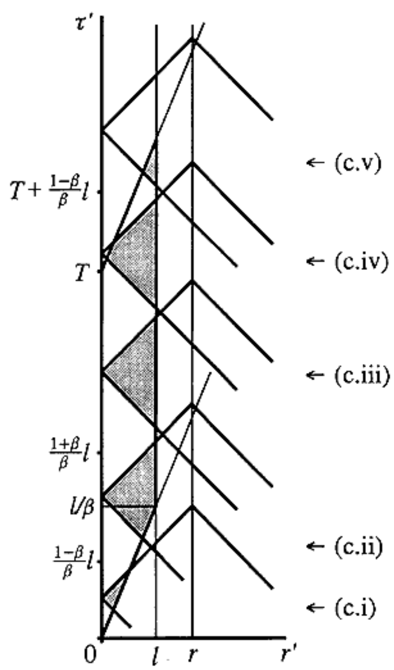

Figure 5. Integration domains on the r', τ' plane for a source current pulse of finite duration T. Instance $r > l$ and $T > \frac{1+\beta}{\beta} l$.

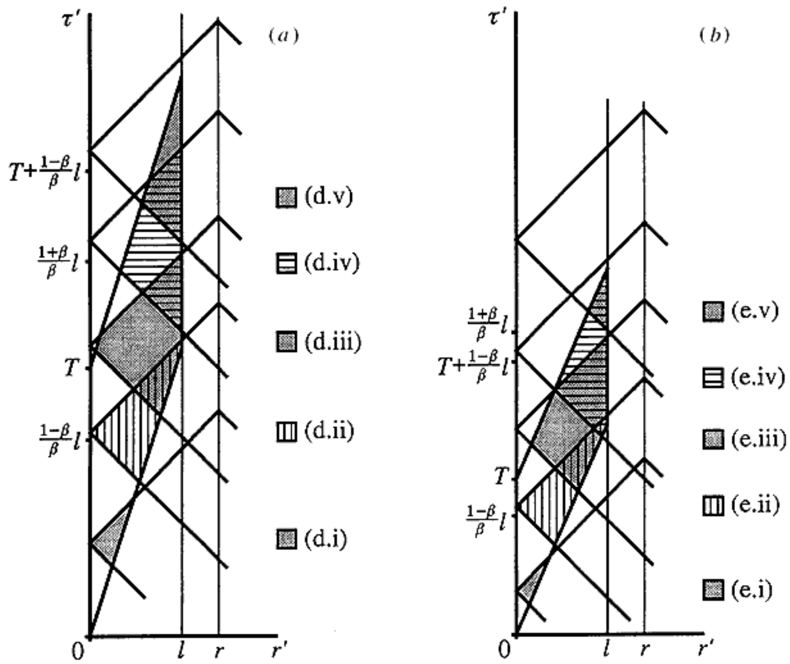

Figure 6. Integration domains for a source current pulse of finite duration. Instance $r > l$ and $\frac{1-\beta}{\beta}l < T < \frac{1+\beta}{\beta}l$: ($a$) case $T > 2l$ and (b) case $T < 2l$.

(c.iii) In the range $\frac{1+\beta}{\beta}l < \tau - r < T$ one should use the expression (20). Note that cases (c.i)–(c.iii) correspond to the situation $\tau - T < r$ in which the finiteness of the source pulse cannot be revealed. Therefore, the expressions obtained for the pulse of infinite duration are still in force.

(c.iv) The first new result appears for the case $T < \tau - r < T + \frac{1-\beta}{\beta}l$, in which the triangle integration domain is truncated by the line $r = \beta(\tau - T)$.

(c.v) The integration domain for the range $T + \frac{1-\beta}{\beta}l < \tau - r < T + \frac{1+\beta}{\beta}l$ is shown on the top of figure 5. For two other possible ranges, $\tau - r < 0$ and $\tau - r > T + \frac{1+\beta}{\beta}l$, the value of I_n is equal zero.

The r', τ'-plane diagrams for the instance $r > l$, $\frac{1-\beta}{\beta}l < T < \frac{1+\beta}{\beta}l$, and $T > 2l$ are represented in figure 6(a).

It is easily seen that for the cases (d.i) $0 < \tau - r < \frac{1-\beta}{\beta}l$ and (d.ii) $\frac{1-\beta}{\beta}l < \tau - r < T$ the finiteness of the source pulse does not manifest itself, so one can use the initial formulae (17) and (19).

A new type of integration domain emerges if (d.iii) $T < \tau - r < \frac{1+\beta}{\beta}l$; here the initial triangle is cut by both lines $\tau' = r'/\beta$ and $\tau' = r'/\beta + T$ corresponding to the front and the backfront of the pulse.

For (d.iv) $\frac{1+\beta}{\beta}l < \tau - r < T + \frac{1-\beta}{\beta}l$ and for (d.v) $T + \frac{1-\beta}{\beta}l < \tau - r < T + \frac{1+\beta}{\beta}l$ we have the same type of the integration domain as in cases (c.iv) and (c.v), respectively.

In figure 6(b) we present the integration domains for similar instance in which $T < 2l$. Here we have the following set of ranges:

(e.i) $0 < \tau - r < \frac{1-\beta}{\beta}l$,

(e.ii) $\frac{1-\beta}{\beta}l < \tau - r < T$,

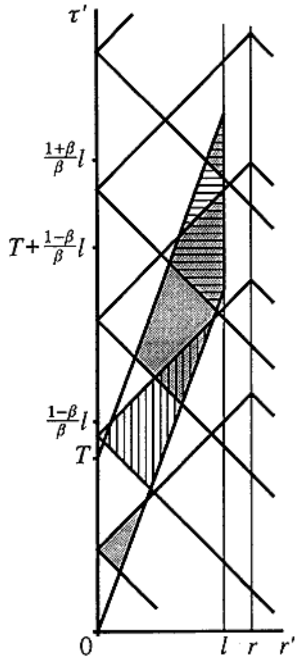

Figure 7. Integration domains for a current pulse of finite duration. Instance $r > l$ and $T < \frac{1-\beta}{\beta}l$.

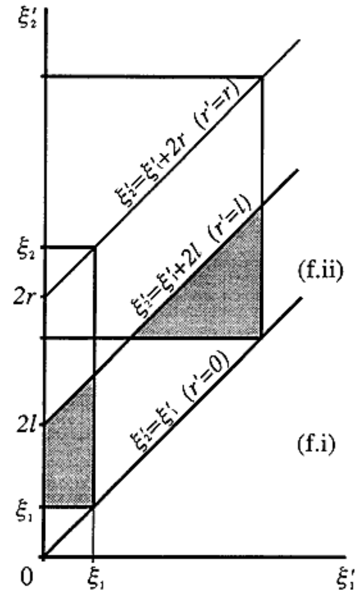

Figure 8. The ξ_1', ξ_2'-plane diagrams for a current pulse of infinite duration moving with the velocity of light, instance $r > l$.

(e.iii) $T < \tau - r < T + \frac{1-\beta}{\beta}l$,
(e.iv) $T + \frac{1-\beta}{\beta}l < \tau - r < \frac{1+\beta}{\beta}l$, and
(e.v) $\frac{1+\beta}{\beta}l < \tau - r < T + \frac{1+\beta}{\beta}l$.

The only new result appears for the case (e.iv) in which the simultaneous application of both front ($\tau' > r'/\beta$) and backfront ($\tau' < r'/\beta + T$) constraints result in the quadrangular integration domain rather than the pentagonal one.

Using the r', τ'-plane diagrams, one can easily obtain the set of integration domains for the cases $2l < T < \frac{1-\beta}{\beta}l$ or $\frac{1-\beta}{\beta}l < T < 2l$—depending on whether $\beta < \frac{1}{3}$ or not—as well as for the case of both $T < 2l$ and $T < \frac{1-\beta}{\beta}l$. The latter case, corresponding to the conditions of so-called 'short' pulse, is illustrated in figure 7.

For the sake of brevity, we do not present corresponding ξ_1', ξ_2'-plane diagrams (which enable us to obtain all the results of this section in the ξ_1, ξ_2-representation) as well as the results for the instance $r < l$ that can be treated analogously.

5. Current pulse moving with the velocity of light

A special case of the problem arises when the source pulse moves with the velocity of light. For the purpose of illustration, here we use the alternative ξ_1', ξ_2'-representation of the integral I_n introduced at the end of section 3 (recall that $\xi_{1,2}' = \tau' \mp r'$).

The integration domains for different observation times to the instance of infinite pulse duration ($T \to \infty$) and $r > l$ are shown in figure 8.

(f.i) For $0 < \xi_1 < 2l$, that is for $0 < \tau - r < 2l$, the integral in question can be written in the form

$$I_n = \tfrac{1}{2}\int_0^{\xi_1} \mathrm{d}\xi_1' \int_{\xi_1}^{\xi_1'+2l} \mathrm{d}\xi_2'\, \Phi_n(\xi_1', \xi_2') \tag{28}$$

where the function $\Phi_n(\xi_1', \xi_2')$ corresponds to $\Phi_n(\tau', r')$ expressed via the new variables

$$\Phi_n(\xi_1', \xi_2') = \frac{4F_n(\xi_1', \xi_2')}{(\xi_1' - \xi_2')^2} P_n\left(1 - 2\frac{(\xi_1 - \xi_1')(\xi_2 - \xi_2')}{(\xi_2' - \xi_1')(\xi_2 - \xi_1)}\right). \tag{29}$$

(f.ii) If $\xi_1 > 2l$, we have the triangle domain of integration and

$$I_n = \tfrac{1}{2}\int_{\xi_1-2l}^{\xi_1} \mathrm{d}\xi_1' \int_{\xi_1}^{\xi_1'+2l} \mathrm{d}\xi_2'\, \Phi_n(\xi_1', \xi_2'). \tag{30}$$

In the case of a source pulse of finite duration and $r > l$ we have two more complicated sets of integration domains (one for $T > 2l$ and another for $T < 2l$).

For $T > 2l$ we have the following set of the parameter interrelations (see figure 9):

(g.i) $0 < \xi_1 < 2l$,

(g.ii) $2l < \xi_1 < T$, and

(g.iii) $T < \xi_1 < T + 2l$.

Again, for the first two cases the finiteness of the pulse cannot be detected, so the expressions (28) and (30) are still valid. In case (g.iii) one has

$$I_n = \tfrac{1}{2}\int_{\xi_1-2l}^{T} \mathrm{d}\xi_1' \int_{\xi_1}^{\xi_1'+2l} \mathrm{d}\xi_2'\, \Phi_n(\xi_1', \xi_2'). \tag{31}$$

For another instance, $T < 2l$, we have the cases shown in figure 10:

(h.i) $0 < \xi_1 < T$, corresponding to the solution (28),

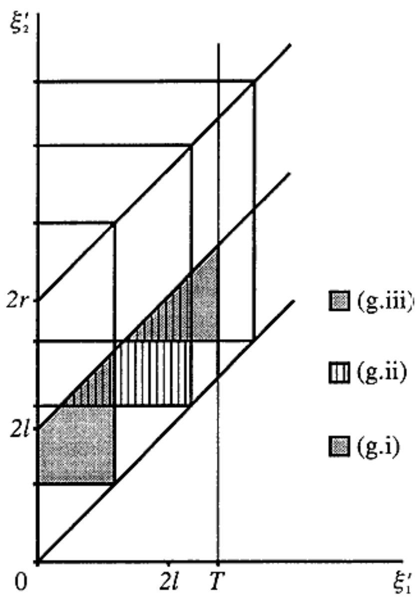

Figure 9. Current pulse of finite duration T moving with the velocity of light, instance $r > l$ and $T > 2l$.

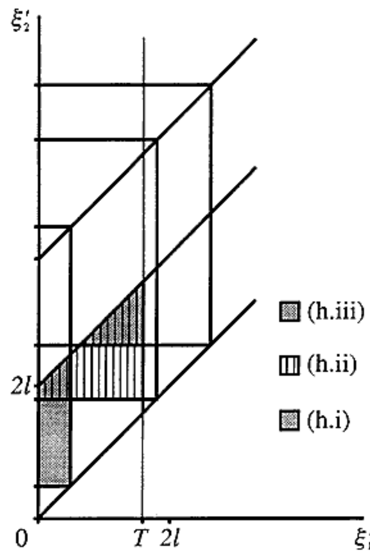

Figure 10. Current pulse moving with the velocity of light, instance $r > l$ and $T < 2l$.

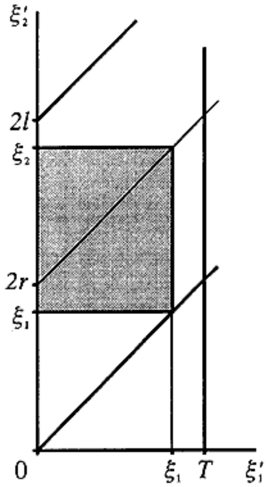

Figure 11. Limiting case (i.i): current pulse of infinite duration moving with the velocity of light along the infinite straight locus ($T \to \infty$ and $l \to \infty$).

(h.ii) $T < \xi_1 < 2l$, where

$$I_n = \tfrac{1}{2} \int_0^T \mathrm{d}\xi_1' \int_{\xi_1}^{\xi_1'+2l} \mathrm{d}\xi_2' \, \Phi_n(\xi_1', \xi_2') \tag{32}$$

(h.iii) $2l < \xi_1 < T + 2l$ with the same domain of integration as in case (g.iii) and, thus, with I_n defined by (31).

For both $T > 2l$ and $T < 2l$ one has $I_n \equiv 0$ if $\xi_1 > T + 2l$.

For the observation locations $r < l$ we can obtain the limits of the integral I_n by the same method. Simple case (i.i) is illustrated in figure 11. Here the integration domain corresponds to the limiting case $T \to \infty$ and $l \to \infty$.

If T and l are finite, constructing the domains of integration for $r < l$ is more complicated than for $r > l$: when the observation point lies inside the source region $0 < r' < l$, the transient processes for each term of the expansions (8) and (15) is defined by the radius of the sphere $r' = r$ rather than that of the sphere $r' = l$. As a consequence, criteria of the integration domain choice include not only T and l (which are parameters), but the radial coordinate r (which is a variable) as well.

We have compared the approach described in sections 3–5 with an alternative method that is analogous to the use of Liénard–Wiechert potentials, and find that the two approaches are consistent with each other. This alternative method will be described in detail in a forthcoming publication [17].

6. Discussion on applications

As an example of the application of the results obtained in the previous sections, here we discuss how to use them for the description of the transient electromagnetic field produced by a moving pulse of line current.

6.1. Delta-pulse of line current as a source of a localized electromagnetic wave

There has been much interest recently in so-called localized waves of both scalar and electromagnetic nature [18–20]. However, two principal issues remain unanswered: (i) what is their physical usefulness and (ii) how it is possible to launch such waves, in particular, how to realize a causal excitation scheme [21, 22]. Earlier we discussed the possible source of a localized wave [23, 24] which is a spike pulsed current moving with the velocity of light along a straight line. The existence of such pulses was proved in the 1960s, when the electromagnetic phenomena accompanying the absorption of x-rays were investigated. Here we discuss peculiarities of the localized wave representation in terms of modes in spherical coordinates.

A simple case of localized wave generation can be observed if we use the source in the form of the delta-pulse of current starting at the moment $\tau = 0$ and moving with the velocity of light along a straight line. In cylindrical coordinates whose origin coincides with the starting point and z axis coincides with the line of propagation of the pulse, the only non-zero component of the current density j_z can be written as

$$j_z = \frac{1}{2\pi}\frac{\delta(\rho)}{\rho}\delta(\tau - z)F(z) \qquad \tau > 0. \tag{33}$$

The choice of the arbitrary function F in the form $F(z) = \exp(-\alpha z)$, where α is a real constant, ensures formation of a localized wave of some definite type. As is shown in [23], here the exact solution to Maxwell's equations yields, for the H_φ-component of the magnetic field strength,

$$\begin{aligned} H_\varphi = -\frac{\partial v}{\partial \rho} &= \frac{1}{4\pi}\delta\left(\tau - \sqrt{\rho^2 + z^2}\right)\frac{\rho}{(\tau - z)\sqrt{\rho^2+z^2}}\exp\left(-\frac{1}{2}\alpha\left(\tau + z - \frac{\rho^2}{\tau - z}\right)\right) \\ &\quad -\frac{1}{4\pi}h\left(\tau - \sqrt{\rho^2+z^2}\right)\frac{1}{\tau - z}\frac{\partial}{\partial\rho}\exp\left(-\frac{1}{2}\alpha\left(\tau + z - \frac{\rho^2}{\tau - z}\right)\right) \\ &= H_{\varphi\delta} + H_{\varphi h} \end{aligned} \tag{34}$$

where the potential v is the solution of the inhomogeneous scalar wave equation (localization property of the wavefunction v for both $\alpha > 0$ and $\alpha < 0$ is investigated in [24]).

On the other hand, the expression (33) can be written in cylindrical coordinates as

$$\begin{aligned} j_r &= \frac{1}{2\pi}\frac{\delta(\cos\vartheta - 1)}{r^2}\delta(\tau - r)\exp(-\alpha r\cos\vartheta) \\ &= \frac{1}{2\pi}\frac{\delta(\cos\vartheta - 1)}{r^2}\delta(\tau - r)\exp(-\tfrac{1}{2}\alpha(\tau + r)) \end{aligned} \tag{35}$$

which corresponds to the representation (1) in which the step functions are replaced by $\delta(\tau - r)$ and $F(\tau, r, \vartheta) = \delta(\cos\vartheta - 1)\exp(-\frac{1}{2}\alpha(\tau + r))$. One can easily check by direct calculation that the coefficients of the expansion (8) for the above current density are given by formula

$$j_n = \frac{n + 1/2}{2\pi r^2}\delta(\tau - r)\exp(-\tfrac{1}{2}\alpha(\tau + r)). \tag{36}$$

Using the method of section 5, case (i.i), we obtain (see figure 11)

$$I_n(\xi_1, \xi_2) = 2\int_{0-}^{\xi_1} \mathrm{d}\xi_1' \int_{\xi_1}^{\xi_2} \mathrm{d}\xi_2' \frac{n + 1/2}{(\xi_2' - \xi_1')^2}\delta(\xi_1')\exp(-\tfrac{1}{2}\alpha\xi_2')P_n\left(1 - 2\frac{(\xi_2 - \xi_2')(\xi_1 - \xi_1')}{(\xi_2' - \xi_1')(\xi_2 - \xi_1)}\right). \tag{37}$$

Upon integration with respect to ξ_1', (37) becomes

$$I_n(\xi_1, \xi_2) = 2(n + \tfrac{1}{2})h(\xi_1)\int_{\xi_1}^{\xi_2} d\xi_2' \frac{1}{\xi_2'^2}\exp(-\tfrac{1}{2}\alpha\xi_2')P_n\left(1 - 2\frac{\xi_1(\xi_2 - \xi_2')}{\xi_2'(\xi_2 - \xi_1)}\right) \tag{38}$$

and for the nth term of the expansion $H_\varphi = \sum_{n=1}^{\infty} H_{\varphi n}$ we have

$$H_{\varphi n} = -\frac{1}{4\pi r} I_n(\xi_1, \xi_2) P_n^1(\cos\vartheta).$$

As the integral I_n is a function of the radial distance r and time τ only, the angular distribution of each term is determined by the factor $P_n(\cos\vartheta)$ (or, in case of the field term, by $P_n^1(\cos\vartheta)$). Hence a single mode $H_{\varphi n}$ does not share the localization property of the spacetime structure of the total field H_φ. This statement holds good provided that summation of expansion (15) yields the correct result. Let us consider it more closely.

Using the solution of scalar problem (5) in which j_r has the form (35) and I_n is defined by (38) and the relation

$$H_\varphi = -\frac{c}{r}\frac{\partial}{\partial\vartheta}\frac{\partial}{\partial\tau}\Pi$$

one can find that

$$H_\varphi = -\frac{1}{2\pi r} h(\xi_1)\frac{\partial}{\partial\vartheta}\sum_{n=1}^{\infty}(n+\tfrac{1}{2})P_n(\cos\vartheta)\int_{\xi_1}^{\xi_2} \mathrm{d}\xi_2' \frac{1}{\xi_2'^2}\exp(-\tfrac{1}{2}\alpha\xi_2')P_n\left(1 - 2\frac{\xi_1(\xi_2-\xi_2')}{\xi_2'(\xi_2-\xi_1)}\right).$$

Interchanging summation and integration and making the substitution

$$\sum_{n=1}^{\infty}(n+\tfrac{1}{2})P_n(\cos\vartheta)P_n\left(1 - 2\frac{\xi_1(\xi_2-\xi_2')}{\xi_2'(\xi_2-\xi_1)}\right) = \delta\left(\cos\vartheta - 1 + 2\frac{\xi_1(\xi_2-\xi_2')}{\xi_2'(\xi_2-\xi_1)}\right)$$

this can be reduced to

$$H_\varphi = -\frac{1}{2\pi r} h(\xi_1)\frac{\partial}{\partial\vartheta}\int_{\xi_1}^{\xi_2} \mathrm{d}\xi_2' \frac{1}{\xi_2'^2}\exp(-\tfrac{1}{2}\alpha\xi_2')\delta\left(\cos\vartheta - 1 + 2\frac{\xi_1}{\xi_2-\xi_1}\frac{\xi_2-\xi_2'}{\xi_2'}\right). \qquad (39)$$

The ξ_2' integration can be performed with the help of relation

$$\delta(\phi(\xi_2')) = \sum_i \delta(\xi_2' - \xi_{2i})\left(\left|\frac{\partial\phi}{\partial\xi_2'}\right|\Bigg|_{\xi_2'=\xi_{2i}}\right)^{-1} \qquad (40)$$

where ϕ is a differentiable function having only simple roots ξ_{2i}. Here they are defined by the equation

$$\phi(\xi_2') = \cos\vartheta - 1 + 2\frac{\xi_1}{\xi_2-\xi_1}\frac{\xi_2-\xi_2'}{\xi_2'} = 0$$

which, for cases $\xi_2' \neq 0$ and $\xi_2 - \xi_1 \neq 0$, gives us the only value lying within the integration domain $\xi_1 < \xi_2' < \xi_2$. Denoting it as ξ_2^0, we have

$$\xi_2^0 = 2\frac{\xi_1\xi_2}{\xi_1+\xi_2-(\xi_2-\xi_1)\cos\vartheta} = \frac{\tau^2-r^2}{\tau - r\cos\vartheta} \qquad (41)$$

which in cylindrical coordinates takes the form

$$\xi_2^0 = \tau + z - \frac{\rho^2}{\tau - z} > 0.$$

From (39), (40), and (41) one has

$$\begin{aligned} H_\varphi &= -\frac{1}{2\pi}\frac{1}{\tau^2-r^2}\frac{\partial}{\partial\vartheta}\exp\left(-\frac{\alpha}{2}\frac{\tau^2-r^2}{\tau-r\cos\vartheta}\right) \\ &= -\frac{1}{4\pi}\alpha\frac{r\sin\vartheta}{(\tau-r\cos\vartheta)^2}\exp\left(-\frac{\alpha}{2}\frac{\tau^2-r^2}{\tau-r\cos\vartheta}\right) \end{aligned}$$

which, in cylindrical coordinates representation, coincides with the second term $H_{\varphi h}$ of (34). Our illustrative consideration turns out to be restricted to the second term $H_{\varphi h}$ of relation (34): representation of the first term $H_{\varphi\delta}$ with the help of the spherical-harmonic expansion is not simple due to the fact that $H_{\varphi\delta} \neq 0$ for $\xi_1 = 0$. However, this example shows that the terms of the spherical-harmonic expansion, when considered separately, do not describe the specific features of the localized fields of the $\exp(\tau + z - \rho^2/(\tau - z))$ family. Only the sum of the terms gives us the description of the localization property.

6.2. Description of the field due to a line current pulse of infinite duration moving with the velocity of light

As is declared in item (iii) of the introduction, we give a simple example of how the representation of the *transient* electromagnetic field in terms of *non-steady-state* spherical harmonics may result in misconception of its spacetime structure. Let the source be the line current pulse of infinite duration moving with the velocity of light whose radial–temporal distribution $f(\tau, r)$ (see comments just below equation (1)) is a function of time reckoned from the wavefront arrival at the observation point, $f(\tau, r) = f(\tau - r)$. Then for the case of infinite current line ($l \to \infty$) we have

$$\begin{aligned} j_r &= \frac{1}{2\pi r^2}\delta(\cos\vartheta - 1)h(\tau - r)f(\tau - r) \qquad \tau > 0 \\ j_r &= 0 \qquad \tau < 0 \end{aligned} \tag{42}$$

the expansion coefficients being

$$j_n = \frac{n + 1/2}{2\pi r^2} h(\tau - r) f(\tau - r).$$

Noticing that the instance in question ($T \to \infty$ and $l \to \infty$) corresponds to the case (i.i) of the previous section, one can obtain the magnetic field representation (15) where

$$I_n = 2(n + \tfrac{1}{2}) \int_{0-}^{\xi_1} \mathrm{d}\xi_1'\, f(\xi_1') \int_{\xi_1}^{\xi_2} \mathrm{d}\xi_2'\, \frac{1}{(\xi_2' - \xi_1')^2} P_n\left(1 - 2\frac{(\xi_2 - \xi_2')(\xi_1 - \xi_1')}{(\xi_2' - \xi_1')(\xi_2 - \xi_1)}\right). \tag{43}$$

Upon changing the variable ξ_2' to $x = 1 - 2(\xi_2 - \xi_2')(\xi_1 - \xi_1')/(\xi_2' - \xi_1')(\xi_2 - \xi_1)$, equation (43) becomes

$$I_n = \frac{n + 1/2}{\xi_2 - \xi_1} \int_{0-}^{\xi_1} \mathrm{d}\xi_1'\, f(\xi_1') \frac{1}{(\xi_1 - \xi_1')(\xi_2 - \xi_1')} \int_{-1}^{1} \mathrm{d}x\, P_n(x). \tag{44}$$

Since $\int_{-1}^{1} \mathrm{d}x\, P_n(x) = 0$ for all values of n except zero, (44) yields $I_n = 0$ for $n \geqslant 1$. Consequently, each term of the expansion $H_\varphi = \sum_{n=1}^{\infty} H_{\varphi n}$, including that for $n = 1$, is equal to zero. We obtain $H_\varphi \equiv 0$ for the arbitrary source-current distribution of the type $f(\tau - r)$, which is obviously incorrect. Actually, this misconception results from condition (6) for $\partial\Pi/\partial\tau$ at the limiting point $r = 0+$. To obtain the correct solution to the problem, one should rewrite the relation for the current density (42) in the form

$$j_r = \frac{1}{2\pi r^2}\delta(\cos\vartheta - 1)h(\tau - r + r_0)f(\tau - r + r_0) \qquad \tau > 0 \tag{45}$$

in which the origin of the spherical coordinates does not coincide with the starting point of the current pulse $r_0 > 0$. Application of the solution method described in section 2 for the case of infinite pulse duration ($T \to \infty$), infinite current line ($l \to \infty$), $r > r_0$, and

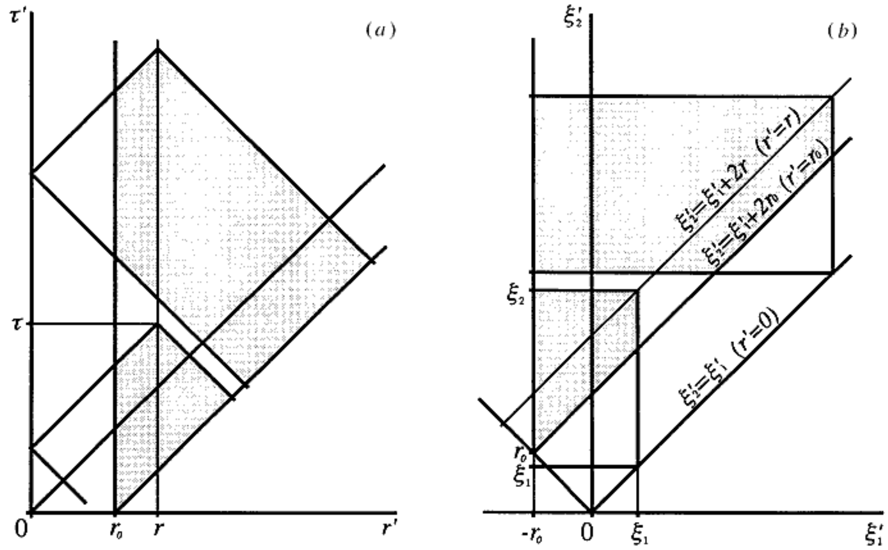

Figure 12. Integration domains in the case of a line current pulse starting at the point r_0 and moving with the velocity of light: (a) on the r', τ' plane and (b) on the ξ_1', ξ_2'-plane.

$\tau - r < r_0$ leads to the diagrams shown in figure 12 and to the following representation of I_n as an iterated integral:

$$I_n = 2\big(n+\tfrac{1}{2}\big)\int_{-r_0}^{\xi_1} \mathrm{d}\xi_1'\, f(\xi_1'+r_0)\int_{\xi_1'+2r_0}^{\xi_2}\mathrm{d}\xi_2'\,\frac{1}{(\xi_2'-\xi_1')^2}P_n\left(1-2\frac{(\xi_2-\xi_2')(\xi_1-\xi_1')}{(\xi_2'-\xi_1')(\xi_2-\xi_1)}\right). \tag{46}$$

Change of variables $\xi_2' \Rightarrow x$ enables us to arrange the integral as

$$I_n = \big(n+\tfrac{1}{2}\big)(\xi_2-\xi_1)\int_{-r_0}^{\xi_1}\mathrm{d}\xi_1'\,\frac{f(\xi_1'+r_0)}{(\xi_1-\xi_1')(\xi_2-\xi_1')}\int_{1-(\xi_1-\xi_1')(\xi_2-\xi_1'-2r_0)/[r_0(\xi_2-\xi_1)]}^{1}\mathrm{d}x\,P_n(x). \tag{47}$$

Here the internal integral is not equal to zero. Hence $H_{\varphi n} \not\equiv 0$ and the general expression for the total magnetic field strength (15) yields

$$\begin{aligned}
H_\varphi &= -\frac{1}{2\pi(\xi_2-\xi_1)}\frac{\partial}{\partial\vartheta}\sum_{n=0}^{\infty}P_n(\cos\vartheta)I_n \\
&= -\frac{2}{c}\frac{\partial}{\partial\vartheta}\sum_{n=0}^{\infty}P_n(\cos\vartheta)(n+\tfrac{1}{2})\int_{-r_0}^{\xi_1}\mathrm{d}\xi_1'\,\frac{f(\xi_1'+r_0)}{(\xi_1-\xi_1')(\xi_2-\xi_1')} \\
&\quad\times\int_{1-(\xi_1-\xi_1')(\xi_2-\xi_1'-2r_0)/[r_0(\xi_2-\xi_1)]}^{1}\mathrm{d}x\,P_n(x) \\
&= -\frac{1}{2\pi}\frac{\partial}{\partial\vartheta}\int_{-r_0}^{\xi_1}\mathrm{d}\xi_1'\,\frac{f(\xi_1'+r_0)}{(\xi_1-\xi_1')(\xi_2-\xi_1')} \\
&\quad\times\int_{1-(\xi_1-\xi_1')(\xi_2-\xi_1'-2r_0)/[r_0(\xi_2-\xi_1)]}^{1}\mathrm{d}x\sum_{n=0}^{\infty}(n+\tfrac{1}{2})P_n(x)P_n(\cos\vartheta).
\end{aligned} \tag{48}$$

Noticing that

$$\sum_{n=0}^{\infty}(n+\tfrac{1}{2})P_n(x)P_n(\cos\vartheta)=\delta(x-\cos\vartheta)$$

one gets

$$\int_{1-(\xi_1-\xi_1')(\xi_2-\xi_1'-2r_0)/[r_0(\xi_2-\xi_1)]}^{1}\mathrm{d}x\sum_{n=0}^{\infty}(n+\tfrac{1}{2})P_n(x)P_n(\cos\vartheta)$$
$$=h\left(\cos\vartheta-1+\frac{(\xi_1-\xi_1')(\xi_2-\xi_1'-2r_0)}{r_0(\xi_2-\xi_1)}\right)$$

and

$$H_\varphi=-\frac{1}{2\pi}\frac{\partial}{\partial\vartheta}\int_{-r_0}^{\xi_1}\mathrm{d}\xi_1'\,\frac{f(\xi_1'+r_0)}{(\xi_1-\xi_1')(\xi_2-\xi_1')}h\left(\cos\vartheta-1+\frac{(\xi_1-\xi_1')(\xi_2-\xi_1'-2r_0)}{r_0(\xi_2-\xi_1)}\right)$$
$$=\frac{1}{2\pi}\sin\vartheta\int_{-r_0}^{\xi_1}\mathrm{d}\xi_1'\,\frac{f(\xi_1'+r_0)}{(\xi_1-\xi_1')(\xi_2-\xi_1')}$$
$$\times\delta\left(\cos\vartheta-1+\frac{(\xi_1-\xi_1')(\xi_2-\xi_1'-2r_0)}{r_0(\xi_2-\xi_1)}\right).$$

The carrier of the δ function is two roots

$$\xi_{1\pm}=\tfrac{1}{2}(\xi_2+\xi_1)-r_0\pm\sqrt{\tfrac{1}{4}(\xi_2-\xi_1)^2+r_0^2-r_0(\xi_2-\xi_1)\cos\vartheta}$$

of the equation

$$\cos\vartheta-1+\frac{(\xi_1-\xi_1')(\xi_2-\xi_1'-2r_0)}{r_0(\xi_2-\xi_1)}=0$$

and for the case in question ($r=\frac{1}{2}(\xi_2-\xi_1)>r_0$, $\tau>0$) only one of them, namely $\xi_1'=\xi_{1-}$, may lie in the domain of integration $[-r_0,\xi_1]$. Using formula (40) and turning to the r,τ-representation, we have the final expression

$$H_\varphi=-\frac{1}{2\pi}h\left(\tau-\sqrt{r^2+r_0^2-2r_0r\cos\vartheta}\right)r_0r\sin\vartheta$$
$$\times\frac{f\left(\tau-\sqrt{r^2+r_0^2-2r_0r\cos\vartheta}\right)}{\sqrt{r^2+r_0^2-2r_0r\cos\vartheta}\left[r^2-\left(r_0+\sqrt{r^2+r_0^2-2r_0r\cos\vartheta}\right)^2\right]}.\tag{49}$$

Remarkably, the correct solution to the electromagnetic problem for the case of source current (42) can be obtained by taking the limit $r_0\to 0$ in equation (49)

$$H_\varphi=\frac{1}{4\pi}h(\tau-r)\frac{\sin\vartheta}{1-\cos\vartheta}\frac{1}{r}f(\tau-r).\tag{50}$$

Notice that this solution is in agreement with the previously published results [25, 26] that have been obtained without use of the multipole expansions.

6.3. Electromagnetic field produced by current instantaneously switched on within a fixed domain

Let the transient source current be switched on at $\tau = 0$ and remain non-zero for all time $\tau > 0$ within the fixed domain $r \in (0, l)$. For this case we do not have a moving current pulse, and the representation of the current density

$$j_r = \frac{1}{2\pi r^2} h(\tau) h(l - r) F(\vartheta) f(\tau, r) \tag{51}$$

does not contain arguments of the type $\tau \pm r$ in the step functions. For this source, the expansion coefficients $\partial \Pi_n / \partial \tau$ of the solution to the scalar problem (5) can be obtained via problem (9) where

$$j_n = \frac{F_n}{2\pi r^2} h(\tau) h(l - r) f(\tau, r). \tag{52}$$

For the case in which the radial coordinate of the observation point r is greater than the radius of the source area l, the integration domains for I_n are shown in figure 13(a) (compare with figures 2(a)–(c) constructed for the moving current pulse). Note that if $\tau - r > l$, we have for I_n the same expression as in case (a.iii) ($\tau - r > \frac{1+\beta}{\beta} l$) of section 3. Due to the finite dimensions of the area in which the current exists at $\tau = 0+$, the solution does not equal zero for $-l < \tau - r < 0$. Here we have

$$I_n = F_n \int_{r-\tau}^{l} \mathrm{d}r' \int_0^{l-(r-\tau)} \mathrm{d}\tau' \, \frac{f(\tau', r')}{r'^2} P_n \left(\frac{r^2 + r'^2 - (\tau - \tau')^2}{2rr'} \right).$$

This situation has no analogues in the cases discussed in the previous sections, which correspond to the source pulses travelling with velocities $0 < \beta \leqslant 1$. Actually, it is the limiting case $\beta \to \infty$ to the solution for $\beta > 1$. To convince oneself that this is correct, it suffices to turn to figure 13(b).

Eventually, let us note that the type of instantaneously switched on current discussed in this subsection is used on frequent occasions as a simplified model of pulsed sources in various problems of electromagnetics (see, for example, [27]).

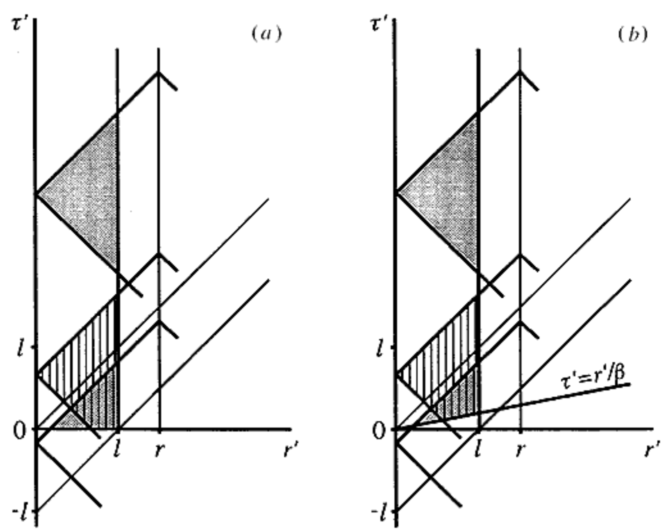

Figure 13. Integration domains in the cases: (a) current instantaneously switched on and (b) current pulse moves with the velocity $\beta > 1$ within the fixed domain $r \in (0, l)$.

7. Conclusion

In the above sections we have considered only axisymmetric solutions. Notably, the results can be extended to the more general non-axisymmetric case. This can be done in the simplest manner for the radial source current, that is, for $\boldsymbol{j} = j(\tau, r, \vartheta, \varphi)\boldsymbol{e}_r$ (TM polarization). Introducing a potential so that

$$\begin{aligned} D_\vartheta &= \frac{1}{r}\frac{\partial^2\Pi}{\partial r\partial\vartheta} \qquad D_\varphi = \frac{1}{r\sin\vartheta}\frac{\partial^2\Pi}{\partial\varphi\partial r} \qquad D_r = -\frac{\partial^2\Pi}{\partial\tau^2} + \frac{\partial^2\Pi}{\partial r^2} \\ H_\vartheta &= \frac{c}{r\sin\vartheta}\frac{\partial^2\Pi}{\partial\tau\partial\varphi} \qquad H_\varphi = -\frac{c}{r}\frac{\partial^2\Pi}{\partial\vartheta\partial\tau} \qquad H_r = 0 \end{aligned} \tag{53}$$

where Π is a scalar function [14], the system of Maxwell's equations is reduced to the scalar inhomogeneous equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) - \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right)\frac{\partial\Pi}{\partial\tau} = \frac{1}{c}j_r. \tag{54}$$

As is seen from (53), one can readily find H_φ and H_ϑ by differentiating the solution of (54) with respect to the angular variables. As previously, calculation of the electric field components involves integration with respect to the time variable.

To separate the angular variables, let us turn to the representations

$$\begin{aligned} \frac{\partial\Pi}{\partial\tau} &= \sum_{n,m}\frac{\partial\Pi_{nm}}{\partial\tau}P_n^m(\cos\vartheta)\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix} \\ j_r &= \sum_{n,m} j_{nm}P_n^m(\cos\vartheta)\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix} \\ \vartheta &\in [0,\pi] \qquad \varphi \in [0, 2\pi] \end{aligned} \tag{55}$$

where

$$P_n^m(\cos\vartheta) \stackrel{\text{def}}{=} \sin^m\vartheta\frac{\partial^m}{(\partial\cos\vartheta)^m}P_n(\cos\vartheta).$$

Substituting (55) in (54) and taking into consideration the initial (3) and boundary conditions (7), we are led to the problem for the expansion coefficients

$$\begin{aligned} &\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial r^2} + \frac{n(n+1)}{r^2}\right)\frac{\partial\Pi_{nm}(\tau,r)}{\partial\tau} = \frac{1}{c}j_{nm}(\tau,r) \\ &\frac{\partial\Pi_{nm}}{\partial\tau} \equiv 0 \qquad \tau < 0 \qquad \left.\frac{\partial\Pi_{nm}}{\partial\tau}\right|_{r=0+} = 0 \end{aligned} \tag{56}$$

in which the differential operator of the equation *does not depend on* m. Thus, the solution to the above problem

$$\frac{\partial\Pi_{nm}}{\partial\tau} = \frac{1}{2c}\iint_{\mathcal{D}}\mathrm{d}\tau'\,\mathrm{d}r'\,j_{nm}(\tau',r')P_n\left(\frac{r^2+r'^2-(\tau-\tau')^2}{2rr'}\right) \tag{57}$$

depends on m *only* through the expansion coefficient of the source j_{nm}. Comparing (57) with the previously obtained result (12), one can see that calculations of the radial–temporal part of the desired solution can be carried out completely in the framework of the solution schemes that have been considered in the present paper.

It should be noted that, for arbitrary temporal variation of the current pulse, the time dependence of the generated wave is different for different angular coordinates of the observation point while each term of the spherical-harmonic series is a function of time and radial coordinate multiplied by a definite function of the angular variables ϑ and φ, see relations (8) and (55). Therefore it is impossible to estimate the field structure on the basis of a finite sum of the spherical-harmonic modes: a single mode or several modes of the expansions do not share the properties of the total field. It is the total sum of the terms that gives us the correct result (see examples of section 6), which can be calculated for several occasions only, as was originally pointed out by Kharkevich [28].

The research presented in this paper does not exhaust the potentialities of description of the transient electromagnetic fields due to a moving pulsed current in terms of modes in spherical coordinates. Here complete consideration has been carried out for the case $r > l$ only, which is more interesting in terms of applications. As shown in subsection 6.2, for some occasions it is necessary to use the results concerned with the current distributions akin to (45) in which the origin of the spherical coordinates does not coincide with the starting point of the current pulse.

Instances $r < l$ and $\beta > 1$ are presented in the form of straightforward examples. It is evident that detailed consideration of these situations can be carried out with the help of the foregoing solution schemes, but this cannot be done within the framework of one paper.

All the preceding concerns the radial current only. However, one can use the results for calculation of fields due to arbitrary source-current density vector. The simplest way to illustrate this is to write equations for the Cartesian components D_i and H_i of the electromagnetic field

$$\begin{aligned} &\frac{\partial^2}{\partial\tau^2}D_i - \nabla^2 D_i = -\frac{1}{c}\left(\frac{\partial}{\partial\tau}\boldsymbol{j} + c\,\mathrm{grad}\,q\right)_i \\ &\frac{\partial^2}{\partial\tau^2}H_i - \nabla^2 H_i = (\mathrm{curl}\,\boldsymbol{j})_i. \end{aligned} \tag{58}$$

The right-hand sides of these equations are defined by the Cartesian components of the current density vector j_i and by the charge density q. Each scalar equation (58), being represented in spherical coordinates, is similar to equation (54). Hence, solutions to these equations for homogeneous initial conditions are defined, in corresponding notation, by formula (57). This enables us to use the results obtained above.

Let us give another example. We represent the field vectors in the form

$$\boldsymbol{H} = c\,\mathrm{curl}\,\frac{\partial}{\partial\tau}\boldsymbol{\Pi} \qquad \boldsymbol{D} = -\frac{\partial^2}{\partial\tau^2}\boldsymbol{\Pi} + \mathrm{grad}\,\mathrm{div}\,\boldsymbol{\Pi}$$

where $\boldsymbol{\Pi}$ is the Hertz vector. Then the function $(\partial/\partial\tau)\boldsymbol{\Pi}$ is a solution of the vector wave equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \nabla^2\right)\frac{\partial}{\partial\tau}\boldsymbol{\Pi} = \frac{1}{c}\boldsymbol{j}$$

and for the *Cartesian components* of the desired function expressed via *variables of the spherical coordinate system* one has equations which are similar to (54). Thus, the solution is again defined by equation (57). Here the components of the magnetic field strength can be obtained explicitly while for calculation of the electric induction one should integrate with respect to the time variable.

Note that the solution for arbitrary current density vector cannot be represented with the help of a one-component vector, which makes the final results difficult to analyse.

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